3.5

Quick answer: If you are in a free-falling elevator, then there is no gravity. With no gravity, everything ~~follows~~ (all projectiles) follows straight paths, Peas, light, everything.

Longer answer: Gravity is causing both you and the pea to accelerate downward at 9.8 m/s^2. Therefore, relative to you, the pea will not accelerate. Motion without acceleration is constant velocity: straight lines, constant speed.

3.6

a) Due to length contraction, the distance between the asteroids will be decreased.

b) The masses will be larger: $m_{rel} = \gamma \cdot m_{rest}$.

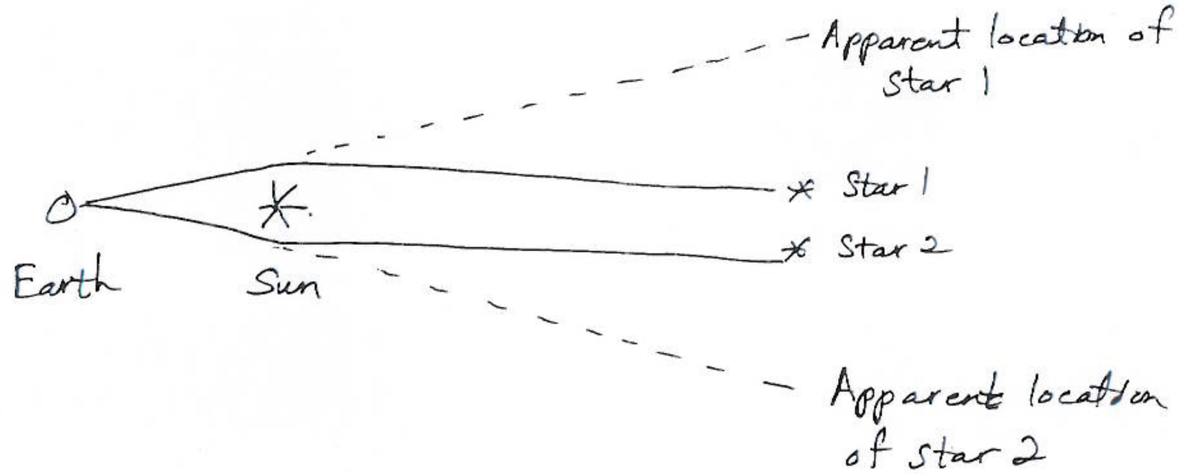
c) Using larger masses and a smaller distance in Newton's formula for gravity $\left(F = G \cdot \frac{m_1 \cdot m_2}{r^2}\right)$ will give a larger value for the force. That's not going to agree with the rocket measurements.

4.1

Where gravity is weak, the predictions of Newton and Einstein are so close you can't tell the difference (within the uncertainty of your measurements).

You need to be in regions of stronger gravity to see the differences. And in our solar system that means close to the Sun. Mercury is the planet closest to the Sun, so it shows the greatest effect.

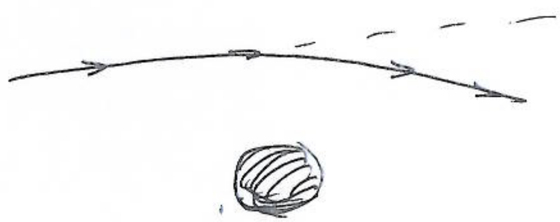
4.2



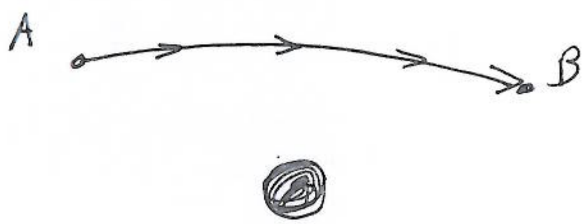
Our sun's presence would make the stars appear to be farther apart.

4.3

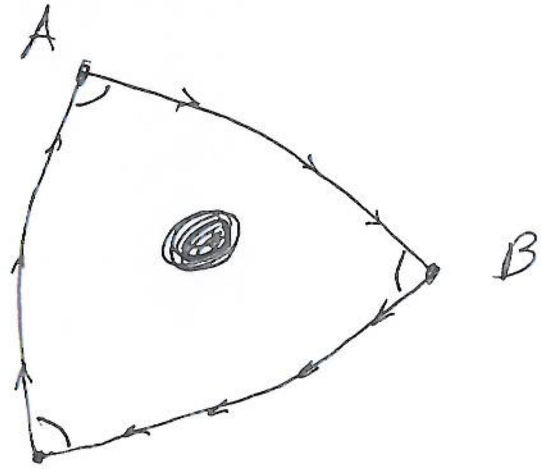
Here's what a single beam of light does as it passes a massive object:



Now I'll put in two space ships; A and B:



What will it look like with C (and B shoots a beam of light toward C)?



The sum of angles A, B, and C will exceed 180°.

4.4

Method 1: Use the formula

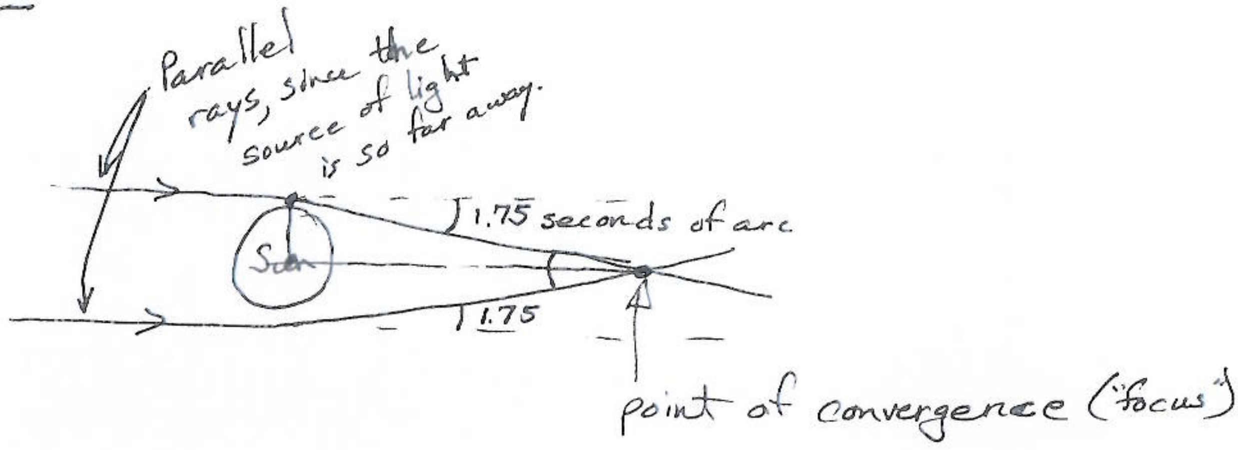
$$\Delta\Theta_E = \frac{4 G M_E}{c^2 R_E} = \frac{4(6.67\times10^{-11})(6\times10^{24})}{(3\times10^{8})^2(6.37\times10^{6})}$$

$$= 2.79\times10^{-9} \text{ radians}$$

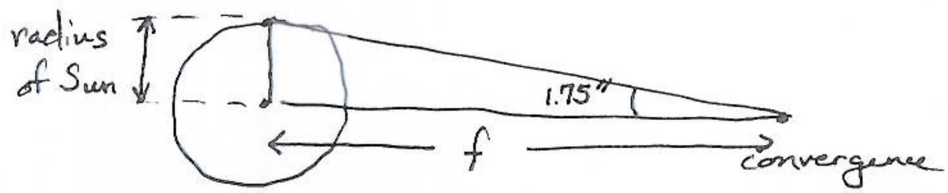
$$= 1.60\times10^{-7} \text{ degrees}$$

$$= 5.76\times10^{-4} \text{ seconds of arc}$$

4.5



There is a triangle we can examine:



Using trigonometry, we can find f.

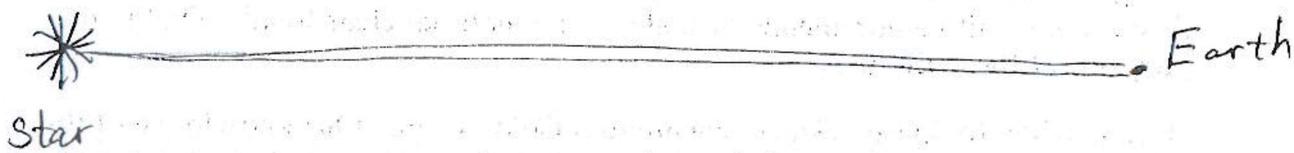
$$\tan(1.75'') = \frac{r_{sun}}{f}, \text{ so}$$

$$f = \frac{r_{sun}}{\tan 1.75''} = \frac{7 \times 10^8 \text{ m}}{\tan\left(\left(\frac{1.75}{3600}\right)^\circ\right)} = \frac{7 \times 10^8 \text{ m}}{8.48 \times 10^{-6}}$$

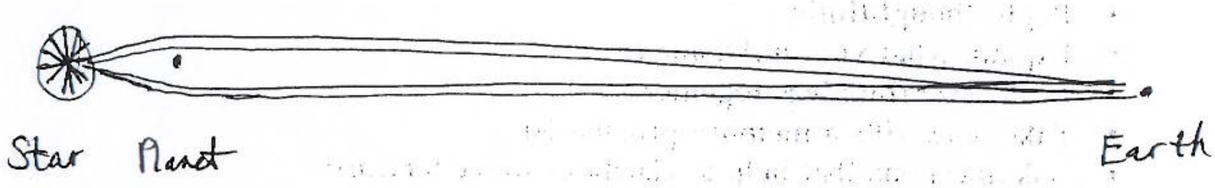
$$f = 8.25 \times 10^{13} \text{ m}$$

This is about 3 light-days, or about 25 times further from the Sun than Pluto orbits.

4.6

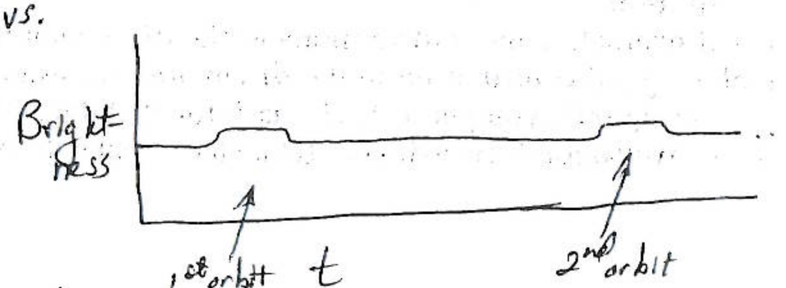


Under normal circumstances, only a small fraction of the light leaving the star will be traveling toward the Earth



With the planet in between, light passing near the planet (along its entire circumference) will get bent enough to head toward the Earth. More light is bent toward the Earth than is blocked by the planet, so the total brightness goes up.

A brightness vs. time graph would look like this:



This has been observed

4.7

We need the gravitational redshift formula for this:

$$f_{at\,\infty} = f_{at\,source} * \sqrt{1 - \frac{2GM}{c^2 r}}$$

~~Not the~~
The fractional change $= \frac{\Delta f}{f} = \frac{f_{source} - f_{\infty}}{f_{source}}$.

$$\therefore \frac{\Delta f}{f} = \frac{\cancel{f_{source}}\left(1 - \sqrt{1 - \frac{2GM}{c^2 r}}\right)}{\cancel{f_{source}}} = 1 - \sqrt{1 - \frac{2GM}{c^2 r}}.$$

For the Sun, $M = 1.99 \times 10^{30}$ kg and $r = 7 \times 10^{8}$ m.

$$\frac{\Delta f}{f} = 1 - 0.9999978931 = 2.1 \times 10^{-6}$$

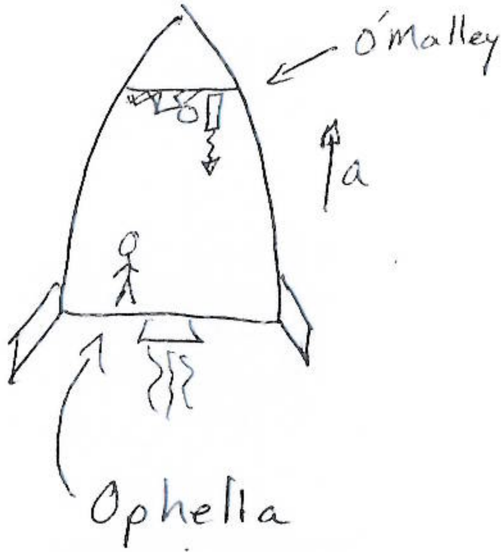
A shift of only 2 parts in a million!

For the Earth, $M = 5.98 \times 10^{24}$ kg and $r = 6.38 \times 10^{6}$ m.

$$\frac{\Delta f}{f} = 1 - 0.9999999993 = 6.9 \times 10^{-10}$$

(That's a ~~$~~ shift of 69 billionths.)

4.8



It takes a little bit of time for the light that leaves the laser to reach Ophelia. She speeds up a bit during that time, but in this case her extra velocity is directed toward the laser rather than away from it. It is as though she is moving toward the source, and so she observes a higher frequency. She observes a "blueshift."

4.89

Cosmological Redshift:

$$f_{observed} \sim \text{as low as } f_{Source} \cdot \frac{1}{4}$$

Gravitational Redshift:

$$f_{observed} = f_{Source} \cdot \sqrt{1 - \frac{2GM}{c^2 r}}$$

Q: How does the square root term compare to $\frac{1}{4}$?

Let's take a star like our own. The light we see is emitted at the surface of the star.

$\therefore$ $M = 2 \times 10^{30}$ kg and $r = 7 \times 10^{8}$ m.

$$\sqrt{1 - \frac{2GM}{c^2 r}} = 0.999997 8825.$$

$\left(\frac{\Delta f}{f} \sim 2 \times 10^{-6} \right)$

For the cosmological case, $\frac{\Delta f}{f} \sim \frac{3}{4}$!)

The redshifts are entirely different in size.

5.1

Let's begin the easy way: assume a stationary Earth.

Mass of Earth

$$\Delta t_{\text{at clock}} = \Delta t_{\text{at } \infty} \cdot \sqrt{1 - \frac{2GM}{c^2 r}}$$

r = your location, which is the surface of the Earth. $\therefore r = 6.38 \times 10^6 \text{ m}$.

$$100 \text{ yrs} = \Delta t_\infty \cdot \underbrace{\sqrt{1 - \frac{2 \cdot (6.67 \times 10^{-11})(6 \times 10^{24} \text{ kg})}{(3 \times 10^8)^2 (6.38 \times 10^6)}}}_{.999\ldots}$$

$$100 \text{ yrs} = \Delta t_\infty \cdot (1 - 7 \times 10^{-10})$$

$$\therefore \Delta t_\infty = \frac{100}{1 - 7 \times 10^{-10}} \simeq 100(1 + 7 \times 10^{-10})$$

$$\Delta t_\infty \simeq 100 \text{ yrs} + 7 \times 10^{-8} \text{ yrs}$$

$$= 100 \text{ yrs} + 2.2 \text{ seconds}.$$

The Earth's gravity is not much help in prolonging your life.

5.2

For clock 1, located at r_1, traveling with speed u_1,

$$T_1 = \frac{T_\infty}{\sqrt{1 - \frac{2GM}{c^2 r_1} - \frac{u_1^2}{c^2}}}$$

For clock 2,

$$T_2 = \frac{T_\infty}{\sqrt{1 - \frac{2GM}{c^2 r_2} - \frac{u_2^2}{c^2}}}$$

$$\left(\therefore T_\infty = T_2 \cdot \sqrt{1 - \frac{2GM}{c^2 r_2} - \frac{u_2^2}{c^2}}\right)$$

Substitute for T_∞ in the top equation:

$$T_1 = \frac{T_2 \sqrt{1 - \frac{2GM}{c^2 r_2} - \frac{u_2^2}{c^2}}}{\sqrt{1 - \frac{2GM}{c^2 r_1} - \frac{u_1^2}{c^2}}}$$, or in neater form,

$$\boxed{T_1 = T_2 \cdot \sqrt{\frac{1 - \frac{2GM}{c^2 r_2} - \frac{u_2^2}{c^2}}{1 - \frac{2GM}{c^2 r_1} - \frac{u_1^2}{c^2}}}}$$

5.3 GPS Satellite

For simplicity's sake, we'll ignore the Sun. (Or more reasonably, the Sun's effect goes away because we are in free fall around the Sun)

$$T_{\substack{\text{clock on}\\ \text{Earth}}} \cdot \sqrt{1 - \frac{2GM_E}{c^2 R_E} - \frac{u_E^2}{c^2}} = T_{\text{clock at } \infty}$$

u = speed of clock on surface of Earth; due to Earth's rotation.

$$T_{\substack{\text{clock on}\\ \text{Satellite}}} \cdot \sqrt{1 - \frac{2GM_E}{c^2(R_E + \text{altitude})} - \frac{u_{sat}^2}{c^2}} = T_{\text{clock at } \infty}$$

$$\therefore T_{\substack{\text{clock on}\\ \text{satellite}}} = T_{\substack{\text{clock}\\ \text{on Earth}}} \cdot \sqrt{\frac{1 - \frac{2GM_E}{c^2 R_E} - \frac{u_E^2}{c^2}}{1 - \frac{2GM_E}{c^2(R+a)} - \frac{u_s^2}{c^2}}}.$$

$G = 6.67 \times 10^{-11}\ N \cdot m^2/kg^2$

$M_E = 6 \times 10^{24}\ kg$

$R_E = 6.37 \times 10^6\ m$

$c = 3 \times 10^8\ m/s$

$a = 4 \times 10^5\ m$

$$u_E \text{ (at equator)} = \frac{\text{Circumference}}{1 \text{ day}} = \frac{2\pi R_E}{86400} = 463\ m/s$$

$u_{sat} = ?$ $F = ma$, so $G\frac{M_E \cancel{M_S}}{(R_E+a)^2} = \frac{u_s^2}{(R_E+a)} \cdot \cancel{M_S}$

$$\therefore u_{sat} = \sqrt{\frac{GM_E}{R_E + a}} = 7700\ m/s$$

$$\text{So, } T_{sat} = T_{Earth} \cdot \frac{0.9999999993}{0.9999999990} = T_E \cdot 1.0000001$$

[illegible] satellite clock is just a bit longer than the Earth clock

5.4

Time is slowed if you are "deeper" into a gravitational field. Time is slowed if you are lower.

∴ If I live on the ground floor, a person in the penthouse would say that my clocks all ran slow, and as a result I lived a bit longer.

Sidebar: Although I really lived longer by the measurements of the person in the penthouse, the length of my life (as measured by my clocks) is unchanged.

5.5

For this problem we want

$$\text{dilation factor}_{\substack{\text{clock on} \\ \text{Equator}}} = \text{dilation factor}_{\substack{\text{clock in} \\ \text{orbit}}}$$

$$\therefore \sqrt{1 - \frac{2GM}{c^2 r_e} - \frac{u_e^2}{c^2}} = \sqrt{1 - \frac{2GM}{c^2 r_o} - \frac{u_o^2}{c^2}}$$

$$\therefore \frac{2GM}{c^2 r_e} + \frac{u_e^2}{c^2} = \frac{2GM}{c^2 r_o} + \frac{u_o^2}{c^2}.$$ ← common factor of $\frac{1}{c^2}$

$$\therefore \frac{2GM}{r_e} + u_e^2 = \frac{2GM}{r_o} + u_o^2$$

From Newtonian physics, we know that the orbital speed squared for an object in a circular orbit is

$$v_o^2 = \frac{GM}{r} \left(= u_o^2 \text{ for this problem}\right).$$

$$\therefore \frac{2GM}{r_e} + u_e^2 = \frac{2GM}{r_o} + \frac{GM}{r_o} = \frac{3GM}{r_o}$$

~~So now we have~~ Also, $u_e = \frac{\text{distance}}{\text{time}} = \frac{2\pi r_e}{86400 \text{ sec}}$.

Now we have $\frac{2GM}{r_e} + \left(\frac{2\pi r_e}{86400}\right)^2 = \frac{3GM}{r_o}$

$r_e = 6.38 \times 10^6$ m; $M = 5.98 \times 10^{24}$ kg; $G = 6.67 \times 10^{-11}$ N·m²/kg².

$$1.25 \times 10^8 = \frac{3GM}{r_o} \rightarrow r_o = \frac{3GM}{1.25 \times 10^8} = 9.5 \times 10^6 \text{ m}$$

... 3.17×10^6 m ~ ½ radius of Earth

For Our Sun,

6.1 $r_{sch} = \dfrac{2 \cdot G \cdot M}{c^2}$

Sun If $M = 2 \times 10^{30}$ kg,

$$r = \frac{2 \cdot (6.67 \times 10^{-11})(2 \times 10^{30} \text{ kg})}{(3 \times 10^8)^2} = \sim 3{,}000 \text{ meters!}$$

(3 km)

Galaxy-center For $M = (2.6 \times 10^6) * (2 \times 10^{30} \text{ kg})$,

$$r = \frac{2(6.67 \times 10^{-11})(2 \times 10^{30})(2.6 \times 10^6)}{(3 \times 10^8)^2} = 7.7 \times 10^9 \text{ meters}$$

(For comparison, the distance from the Earth to the sun is 1.5×10^{11} meters.)

Earth $M = 6 \times 10^{24}$ kg,

$$r = \frac{2(6.67 \times 10^{-11})(6 \times 10^{24})}{(3 \times 10^8)^2} = 0.00889 \text{ m}$$

(8.89 mm)

6.2

$$f_{at\,\infty} = f_{at\,source} \cdot \sqrt{1 - \frac{2GM}{c^2 r}}$$

If $r = \frac{3GM}{c^2}$, $\frac{2GM}{c^2 r} = \frac{2\cancel{G}\cancel{M}}{\cancel{c^2}} \cdot \frac{\cancel{c^2}}{3\cancel{G}\cancel{M}} = \frac{2}{3}$.

$$\therefore f_{at\,\infty} = f_{at\,source} \cdot \sqrt{1 - \frac{2}{3}}$$

$$= 1 \times 10^{15}\,\mathrm{Hz} \cdot \sqrt{\frac{1}{3}}$$

$$= 5.77 \times 10^{14}\,\mathrm{Hz}.$$

This is green light.

6.3

$$f_{at\,\infty} = f_{at\,source} * \sqrt{1 - \frac{2GM}{c^2 r}}$$

If $r = \frac{2GM}{c^2}$, $\frac{2GM}{c^2 r}$ becomes $\frac{2GM}{c^2\left(\frac{2GM}{c^2}\right)} = 1$.

$$\therefore f_{at\,\infty} = f_{at\,source} * \sqrt{1-1} = 0.$$

Significance: If the frequency is zero, it isn't really a wave anymore. There is nothing to see. But $r = \frac{2GM}{c^2}$ is the Schwarzschild radius, so this source of light is right at the edge of a black hole – nothing should get out.

6.4

The Earth has much less gravitational pull than a black hole, so we will consider the Earth to be at infinity relative to the black hole. We will assume that all velocities are zero.

Time dilation:

$$\underset{\substack{\uparrow \\ \frac{1}{12}\text{ year}}}{\Delta t_{\text{at clock}}} = \underset{\substack{\uparrow \\ 1\text{ year}}}{\Delta t_{\text{at } \infty}} \cdot \sqrt{1 - \frac{2GM}{c^2 r}} \quad \left(= \Delta t_{\infty} \cdot \sqrt{1 - \frac{r_{sch}}{r}} \right.$$

To get the answer as a number times the Schwarzschild radius

solve: $\frac{1}{12}$ year $= 1$ year $\cdot \sqrt{1 - \frac{r_{sch}}{r}}$.

$$\therefore \frac{1}{144} = 1 - \frac{r_{sch}}{r} \rightarrow \frac{r_{sch}}{r} = \frac{143}{144} \rightarrow \frac{r}{r_{sch}} = \frac{144}{143}.$$

$$\therefore \boxed{r = \frac{144}{143} * r_{sch}}$$

In meters:

$$\frac{1}{12} = 1 \cdot \sqrt{1 - \frac{2(6.67\times10^{-11})(8\times10^{30})}{(3\times10^{8})^2\ r}} = \sqrt{1 - \frac{11858}{r}}$$

$$\frac{1}{144} = 1 - \frac{11858}{r} \rightarrow \frac{11858}{r} = \frac{143}{144} \rightarrow r = \frac{144 \cdot 11858}{143} = \boxed{11,941 \text{ meters}}$$

6.5

The time dilation factor is $\sqrt{1-\frac{2GM}{c^2 r}} = \sqrt{1-\frac{r_{sch}}{r}}$.

If $r = r_{sch}(1+\delta)$, $\frac{r_{sch}}{r}$ becomes $\frac{r_{sch}}{r_{sch}(1+\delta)} = \frac{1}{1+\delta}$.

$\therefore$ time factor $= \sqrt{1 - \frac{1}{1+\delta}}$.

MacLaurin Series: $\frac{1}{1+\delta} = (1+\delta)^{-1} \cong 1-\delta$.

$\therefore$ time factor $\cong \sqrt{1-(1-\delta)} = \sqrt{\delta}$.

But what is δ?

$$r = r_{sch}(1+\delta), \text{ so } 1+\delta = \frac{r}{r_{sch}}.$$

$$\therefore \delta = \frac{r}{r_{sch}} - 1 = \frac{r - r_{sch}}{r_{sch}} = \frac{\text{altitude}}{r_{sch}}.$$

$\therefore$ time factor $\cong \sqrt{\frac{\text{altitude}}{r_{sch}}}$.

Note: This was used to get the correct relationship between O'Malley and Ophelia's locations and the relative time dilation.

6.6

The black hole loses mass at a rate of $\left(\frac{hc^4}{30720\pi^2 G^2 M^2}\right)$ kg/sec.

$$\frac{hc^4}{30720\pi^2 G^2} \simeq 4\times10^{15}\ \frac{kg^3}{s}$$

$\therefore$ Rate $\frac{4\times10^{15}}{M^2}$.

If $M = 2\times10^{30}$ kg, rate $= \frac{4\times10^{15}}{(2\times10^{30})^2} = 10^{-45}$ kg/sec.

If $M = 6\times10^{36}$ kg, rate $= \frac{4\times10^{15}}{(6\times10^{36})^2} = 1.1\times10^{-58}$ kg/sec.

Wow! That's tiny!

6.7

Begin with the formula for mass loss:

$$\frac{dM}{dt} = \frac{-4\times10^{15}}{M^2}.$$

To find what happens to mass over time, we need to replace the derivative by finite differences. This will introduce some inaccuracy but we can get an approximate answer anyway.

$$\frac{\Delta M}{\Delta t} = \frac{-4\times10^{15}}{M^2}$$

To put this on a TI calculator, we need to use the language of TI. First, put your calculator in "SEQ" mode (sequence).

$\Delta M = M_2 - M_1$ or $M_3 - M_2$ or $M_4 - M_3$, etc, and $\Delta M = (-4\times10^{15})/M^2 * \Delta t$. For ease, we will take each step in time (Δt) to be 1 second.

M will be represented by the sequence variable u.

$$\therefore\ u(n) = u(n-1) - 4E15/u(n-1)^2$$

This formula calculates the mass (u) at any time (n) based on the value of the mass one second earlier (n−1)

Continued →

When you hit the "y =" button, you can enter this formula. You will also enter

$\eta \text{Min} = 0$ and $u(\eta \text{Min}) = \{1E6\}$.

This means the clock begins at $t(\eta) = 0$, and the mass will initially be 10^6 kg.

Next, go to "TBLSET" and select

TblStart = 0 and ΔTbl = 1.

This will start the table listing at $t = 0$ and proceed one second at a time.

Now go to "TABLE." Here's what I have:

η	$u(\eta)$
0	1E6
1	996000
2	991968
3	987903
4	983804
5	979671
6	975504

You can see the mass is decreasing as time goes by.

Pushing the 'down' cursor button will calculate new rows, one at a time.

By 40 seconds we are down to 800,000 kg, roughly, and the rate of evaporation is increasing. At 85 seconds, I have ~123,000 kg, and at 86 seconds the mass goes negative.

∴ The lifetime is roughly 85 or 86 seconds.

Note: By using a smaller Δt you can get a more accurate result.

By using the formula from problem 6.8 we find that the exact answer for this initial mass is $83.\overline{3}$ seconds. Not too different from the approximate answer found above.

6.8

$$\frac{dM}{dt} = (-4 \times 10^{15}) \cdot M^{-2}$$

$$\therefore \quad M^2 \, dM = (-4 \times 10^{15}) \, dt$$

Let's integrate this!

$$\int_{M_i}^{M_f} M^2 \, dM = (-4 \times 10^{15}) \int_{t_i}^{t_f} dt$$

M_i is the initial mass of the black hole. If it evaporates completely, $M_f = 0$.

$$\frac{1}{3} M^3 \Big|_{M_i}^{M_f \to 0} = -4 \times 10^{15} (t) \Big|_{t_i}^{t_f}$$

$$-\frac{M_i^3}{3} = -4 \times 10^{15} \, \Delta t$$

$$\boxed{\Delta t = \frac{M_i^3}{12 \times 10^{15}} = \frac{M_i^3}{1.2 \times 10^{16}}}$$

Solar Mass? $M_i = 2 \times 10^{30}$ kg, so

$$\Delta t = \frac{8 \times 10^{90}}{1.2 \times 10^{16}} = 6.7 \times 10^{74} \text{ seconds} \simeq 2 \times 10^{67} \text{ years}$$

($\approx 1.5 \times 10^{57}$ times the age of the universe!)

6.9

a) If you add energy, you add mass.

b) If the mass goes up, the rate of energy being emitted drops.

c) A lower rate of radiation corresponds to a lower temperature.

d) This is odd because we've added energy and the temperature dropped!

7.1

The observable universe is the part of the universe within range of our telescopes. Why can't we see it all? Because it takes time for the light of distant stars to reach us. Since the universe is 13.7 billion years old, the edge of the observable universe can be no further than 13.7 light years away.

So the edge isn't the limit of the universe, it's just the limit of what we can see from here. If I could be suddenly transported to the edge of the observable universe, I'd see the same type of stuff I see from here. I would not be at a physical edge.

7.2

If everything expanded in the same way, our meter sticks would not work to measure the expansion. You could never know.

In our own universe, maybe it's just that everything is shrinking and space is staying the same!

7.3

If space stayed the same but atoms were getting smaller, my measuring tools would all be shrinking. As a result, the measured distance to another galaxy would seem to be increasing. It would look just like what we see right now!

We can't tell the difference between shrinking atoms and expanding space.

7.4

Where there is enough matter, space does not expand. Where the density of matter is too low, there isn't enough mass to hold things together and we see expansion.

Result? Galaxies don't expand, but the space between galaxies does.

Note: If Einstein's cosmological constant were to <u>grow</u> (not be constant) then the galaxies might begin to expand while planetary systems could be stable. If it grew more, planetary systems would expand while planets would still hold their own. If it continued to grow....
(But we don't know what the cosmological constant represents, so we don't know what it's going to do. It seems to be a constant for now.)

7.5

If my telescope could see back 13.7 billion years, I could see the beginning of the universe. If the universe had zero size, that means there would only be one point to see, no matter what direction I point my telescope.

7.6

a) $v = Hx$; $v = \frac{dx}{dt}$

$\therefore \frac{dx}{dt} = Hx \longrightarrow \frac{dx}{x} = H\,dt$

Integrate this:

$$\int_{x_i}^{x_f} \frac{dx}{x} = \int_{t_i}^{t_f} H\,dt$$

$$\ln x_f - \ln x_i = H \cdot \Delta t$$

$$\ln\left(\frac{x_f}{x_i}\right) = H \cdot \Delta t$$

Take the exponential of both sides:

$$\frac{x_f}{x_i} = e^{H \cdot \Delta t}$$

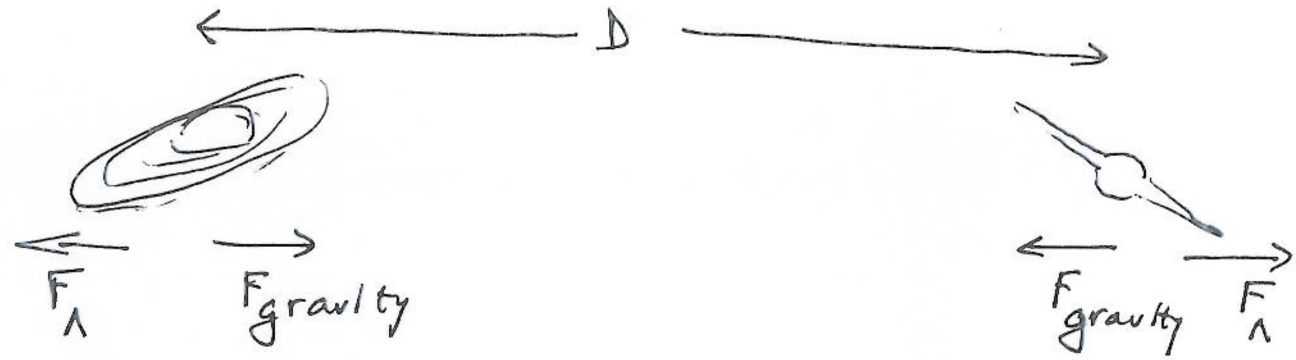
$$\therefore \boxed{x_f = x_i\, e^{H \cdot \Delta t}}$$

b) If $x_f = x_i \cdot e^{H\Delta t}$, $v = \frac{dx_f}{dt} = H x_i\, e^{H \cdot \Delta t}$.

If Δt gets big enough, $v > c$ is going to happen!

7.7

Let's think about two galaxies:



We assume that at distance D, $F_g = F_\Lambda$.

If I move these galaxies closer together, $F_{gravity}$ will increase while F_Λ will decrease.

$\therefore$ The net force will now be attractive, pulling the galaxies even closer together.

If I increase the separation, F_Λ will overcome $F_{gravity}$ and the net force will be repulsive, pushing them even further apart.

This is like a ball on top of a dome - any slight deviation from the center, and the ball rolls away, faster and faster.

The equilibrium is unstable.

7.8

Let's begin with the equation for the universe that Einstein found:

$$\left(\frac{dR}{dt}\right)^2 = \frac{8}{3}\pi G\rho R^2 .$$

The Hubble constant isn't in this equation, but it is in the equation

$$v = H \cdot \text{distance}.$$

v = rate of change of distance = $\frac{dR}{dt}$,

and distance = R.

$\therefore$ the Hubble equation is

$$\frac{dR}{dt} = H \cdot R .$$

$$\therefore H = \left(\frac{dR}{dt}\right) \Big/ R .$$

Now go back to Einstein:

$$\frac{\left(\frac{dR}{dt}\right)^2}{R^2} = \frac{8}{3}\pi G\rho, \text{ so } H = \sqrt{\frac{8}{3}\pi G\rho}.$$

The Hubble constant is proportional to $\sqrt{\text{density}}$.